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Meshless schemes for unsteady Navier–Stokes equations in vorticity formulation using radial basis functions

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Abstract

In this paper, we provide a new scheme for unsteady incompressible flows in vorticity-stream function formulation. Combined with the radial basis functions method, it is an efficient meshless method. Optimal accuracy can be achieved using this method. The efficiency and accuracy are demonstrated by numerical examples.

Keywords: Incompressible flow; Navier–Stokes equations; Vorticity-stream function; RBF; Meshless

1. Introduction

The primitive variable formulation of the incompressible Navier–Stokes equations (NSE) on a domain $\Omega \subset \mathbf{R}^2$ (or $\mathbf{R}^3$) takes the form

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u},$$
$$\nabla \cdot \mathbf{u} = 0, \tag{1.1}$$

where $\mathbf{u} = (u, v)^{\mathrm{T}}$ (or $\mathbf{u} = (u, v, w)^{\mathrm{T}}$), p, and ν are the velocity field, pressure, and kinematic viscosity, respectively. For now we consider the simplest physical boundary condition for $\mathbf{u}$, the no-penetration, no-slip condition

$$\mathbf{u}|_{\Gamma} = 0, \tag{1.2}$$

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where $\Gamma = \partial\Omega$. Adding inhomogeneous terms to the boundary condition only amounts to minor changes in what follows.

There are numerous ways to discretize the unsteady incompressible Navier–Stokes equations (1.1) and (1.2) in time. Undoubtedly, the most popular one consists of using projection methods. This class of techniques has been introduced by Chorin and Temam [5,6,29]. They are time marching algorithms based on a fractional step technique that may be viewed as a predictor–corrector strategy aiming at uncoupling viscous and incompressibility. In recent years, many accurate numerical methods have been provided. For example, some useful numerical methods can be found in [1,10,11,14]. At the same time, many high-order computation methods for Eqs. (1.1) and (1.2) have also been advanced; for details refer to [2,3,7,17,18,30,32,35].

For the 2D case, the vorticity-stream formulation of the NSE reads

$$\begin{aligned} &\partial_t \omega + (\mathbf{u}\cdot\nabla)\omega = \nu\Delta\omega, \\ &\Delta\psi = \omega \end{aligned} \tag{1.3}$$

with no-slip boundary condition

$$\psi|_\Gamma = 0, \quad \left.\frac{\partial\psi}{\partial n}\right|_\Gamma = 0 \tag{1.4}$$

and the velocity is given by

$$\mathbf{u} = \nabla^\perp\psi, \quad \nabla^\perp = (-\partial_y, \partial_x)^{\mathrm{T}}. \tag{1.5}$$

There are numerous ways to discretize Eqs. (1.3) and (1.4) (see [19–21,31,33,34]). In this paper, we will provide a new scheme for the unsteady incompressible Navier–Stokes equations in the vorticity-stream function formulation (1.3) and (1.4). Combined with radial basis functions (RBFs) method, we find that it is an efficient method because it is meshless, accurate and capable of computation in any domain (unlike finite difference method, finite element method and spectral method). Furthermore, the indirect radial basis function network (IRBFN) method (see [4,23–25]) can be used to make our method more accurate.

The outline of this paper is as follows: in Section 2 we simply introduce some elementary knowledge about RBFs. In Section 3 our new scheme will be described and in Section 4 we will presents numerical results to indicate the accuracy and the efficiency of our meshless method. Section 5 contains conclusions.

2. Basic knowledge about radial basis functions

In this section, we simply introduce some elementary knowledge about radial basis functions; for more details refer to [8,15,22,26,27,36,38].

In the last decade, there have been some advanced developments in applying the radial basis functions (RBFs) for the numerical solutions of various types of partial differential equations (PDEs). The initial development was due to the pioneering work of Kansa [15] who directly collocated the RBFs for the approximated solutions of the equations. The Kansas method has been applied successfully to obtain numerical solutions of various types of ordinary and partial differential equations including the biphasic mixture model for tissue engineering problems [12], heat transfer [42], 1D and 2D nonlinear Burgers problems arising from the American option pricing [13]. This method is truly meshless in the sense that

the collocation points can be chosen freely. This definitely removes the complicated meshing problem in using the traditional finite element method (FEM) and finite volume method (FVM), which is a very time-consuming portion of overall computation. Typically, more than 70 percent of overall computing time is spent by mesh generators [9]. The existence, uniqueness, and convergence proofs in applying the RBFs are given in [22,26,27] for scattered data interpolation and recently in [8,36,38] for solving PDEs, respectively. In their papers, two important features of the RBFs method had been observed:

(1) it is a truly meshless algorithm; and
(2) it is spatial dimension independent which can easily be extended to solve high-dimensional problems.

The commonly used RBFs are multiquadric (MQ), Gaussian, thin-plate splines, and compactly supported RBFs. The MQ has been found to provide the most accurate approximation in most of the applications of the RBFs.

The process of interpolation using a radial basic function is as follows: if we are supplied with a finite set of interpolation points $X \subset R^d$ and a function $f : X \to R$, we can construct an interpolant to f of the form

$$(Sf)(x) = \sum_{x_i \in X} \lambda_i \phi(\|x - x_i\|) + p(x) \quad \text{for } x \in R^d, \tag{2.1}$$

where $\|\cdot\|$ denotes the Euclidean distance, and $\phi(r)$ is a radial function. Of course, for Sf to interpolate f the real numbers λ_i and the polynomial p must be chosen to satisfy the system

$$\begin{aligned} &(Sf)(x_i) = f(x_i) \quad \text{for } x_i \in X, \\ &\sum_{x_i \in X} \lambda_i q(x_i) = 0 \quad \text{for all } q \in \pi_{m-1}^d, \end{aligned} \tag{2.2}$$

where π_{m-1}^d denotes all polynomials on R^d of total degree at most $m-1$. From [27,22,38] we know that we have a unique interpolatant $(Sf)(x)$ of f if $\phi(r)$ is a conditional positive-definite radial basis function of order m. The most prominent examples of conditional positive-definite radial basis functions of order m on R^d are

$$\begin{aligned} &\phi(r) = (-1)^{[\beta/2]} r^\beta, \quad \beta > 0, \ \beta \notin 2N_0, \ m \geqslant [\beta/2], \\ &\phi(r) = (-1)^{k+1} r^{2k} \log(r), \quad k \in N, \ m \geqslant k+1, \\ &\phi(r) = (c^2 + r^2)^{\beta/2}, \quad \beta < 0, \ m \geqslant 0, \\ &\phi(r) = (-1)^{[\beta/2]} (c^2 + r^2)^{\beta/2}, \quad \beta > 0, \ \beta \notin 2N_0, \ m \geqslant [\beta/2], \\ &\phi(r) = \mathrm{e}^{-\lambda r^2}, \quad \lambda > 0, \ m \geqslant 0, \\ &\phi(r) = (1-r)_+^4 (1+4r), \quad d \leqslant 3, \ m \geqslant 0. \end{aligned}$$

See e.g. [28] for a comprehensive derivation of the properties of these functions. It is customary to scale a radial basis function $\phi(r)$ by going over to $\phi(r/\alpha)$ with a positive value α, that is roughly proportional to the distance between "neighbouring" data locations. In particular, for the Wendland function $\phi(r) = (1-r)_+^4(1+4r)$ the scaled function has support α. The convergence proofs in applying the RBFs for

scattered data interpolation were given in [37]. For the method of solving partial difference equations, one can see the collocation method in [15], Galekin method in [36], IRBFN method in [4], subdomain or boundary integral method in [40], and Petrov–Galerkin method in [39].

3. Meshless scheme for unsteady Navier–Stokes equations in vorticity formulation using RBFs

Eq. (1.3) can be discretized in time either by an implicit method, a semi-implicit method or an explicit method. One family of semi-implicit methods of order r is obtained by extrapolating $(\mathbf{u}\cdot\nabla)\omega$ from time t^{n-r+1} up to t^n with an $(r-1)$th order polynomial to t^{n+1} and applying the rth order backward differentiation formula (BDFr) to the linear terms to arrive at

$$\sum_{j=0}^{r}\alpha_j\omega^{n+1-j}+\Delta t\sum_{j=1}^{r}\beta_j(\mathbf{u}^{n+1-j}\cdot\nabla)\omega^{n+1-j}-\Delta t\nu\Delta\omega^{n+1}=0. \tag{3.1}$$

The coefficients α_j are given by BDFr and the coefficients β_j are defined by the extrapolation

$$\omega^{n+1}=\sum_{j=1}^{r}\beta_j\omega^{n+1-j}+\mathcal{O}(\Delta t^r). \tag{3.2}$$

The coefficients in (3.1) up to order $r=4$ are given in Table 1.

Another class of time discretizations, based on Adams methods, is derived by integrating (1.3) in time

$$\omega^{n+1}-\omega^n=\int_{t^n}^{t^{n+1}}\nu\Delta\omega\,\mathrm{d}t-\int_{t^n}^{t^{n+1}}(\mathbf{u}\cdot\nabla)\omega\,\mathrm{d}t. \tag{3.3}$$

The integral with the nonlinearity is approximated by an explicit Adams–Bashforth method and the integral with the linear integrand by an implicit Adams–Moulton method. The resulting formula of order r is

$$\omega^{n+1}-\omega^n=\Delta t\sum_{j=0}^{r-1}\beta_j^1\nu\Delta\omega^{n+1-j}-\Delta t\sum_{j=1}^{r}\beta_j^2(\mathbf{u}^{n+1-j}\cdot\nabla)\omega^{n+1-j}. \tag{3.4}$$

The coefficients for $2\leqslant r\leqslant 4$ are given in Table 2.

A common combination is the Adams–Bashforth method of second order and the implicit trapezoidal (or Crank–Nicolson) method [16]. Certainly the last integral in (3.3) can be easily evaluated by an explicit RK scheme at the cost of additional calculations of $(\mathbf{u}\cdot\nabla)\omega$ in interior stages between t^n and t^{n+1}.

Table 1
Coefficients for BDF methods (3.1)

	α_0	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
BDF1	1	−1				1			
BDF2	3/2	−2	1/2			2	−1		
BDF3	11/6	−3	3/2	−1/3		3	−3	1	
BDF4	25/12	−4	3	−4/3	1/4	4	−6	4	−1

Table 2
Coefficients for Adams methods (3.4)

	β_0^1	β_1^1	β_2^1	β_3^1	β_1^2	β_2^2	β_3^2	β_4^2
Adams2	1/2	1/2			3/2	−1/2		
Adams3	5/12	8/12	−1/12		23/12	−16/12	5/12	
Adams4	9/24	19/24	−5/24	1/24	55/24	59/24	37/24	−9/24

Either (3.1) or (3.4) can be written as

$$(1-\alpha\Delta t\nu\Delta)\omega^{n+1}=f_r^n+\Delta t g_r^n, \tag{3.5}$$

where

$$\alpha=\begin{cases}\dfrac{1}{\alpha_0} & \text{in BDF methods,}\\ \beta_0^1 & \text{in Adams methods,}\end{cases}$$

$$f_r^n=f_r^n(\omega^{n+1-r},\ldots,\omega^n)=\begin{cases}-\dfrac{1}{\alpha_0}\displaystyle\sum_{j=1}^{r}\alpha_j\omega^{n+1-j} & \text{in BDF methods } (r\geqslant 1),\\ \omega^n & \text{in Adams methods } (r\geqslant 2),\end{cases}$$

$$g_r^n=g_r^n(\omega^{n+1-r},\ldots,\omega^n,\mathbf{u}^{n+1-r},\ldots,\mathbf{u}^n)$$
$$=\begin{cases}-\dfrac{1}{\alpha_0}\displaystyle\sum_{j=1}^{r}\beta_j(\mathbf{u}^{n+1-j}\cdot\nabla)\omega^{n+1-j} & \text{in BDF methods } (r\geqslant 1),\\ \displaystyle\sum_{j=1}^{r-1}\beta_j^1\nu\Delta\omega^{n+1-j}-\sum_{j=1}^{r}\beta_j^2(\mathbf{u}^{n+1-j}\cdot\nabla)\omega^{n+1-j} & \text{in Adams methods } (r\geqslant 2).\end{cases}$$

After the time discretization, Eq. (1.3) can be written as follows:

$$\begin{aligned}&(1-\alpha\Delta t\nu\Delta)\omega^{n+1}=f_r^n+\Delta t g_r^n,\\ &\Delta\psi^{n+1}=\omega^{n+1}.\end{aligned} \tag{3.6}$$

with the no-slip boundary condition

$$\psi^{n+1}=0,\quad \frac{\partial\psi^{n+1}}{\partial n}=0. \tag{3.7}$$

Due to (3.6) and (3.7), we obtain the Poisson equation of the stream function ψ^{n+1}:

$$\begin{aligned}&(1-\alpha\Delta t\nu\Delta)\Delta\psi^{n+1}=f_r^n+\Delta t g_r^n,\\ &\psi^{n+1}|_\Gamma=0,\quad \left.\frac{\partial\psi^{n+1}}{\partial}\right|_\Gamma=0.\end{aligned} \tag{3.8}$$

The velocity $\mathbf{u}^{n+1}$ and the vorticity ω^{n+1} are given by

$$\mathbf{u}^{n+1} = \nabla^{\perp}\psi^{n+1}, \quad \omega^{n+1} = \Delta\psi^{n+1}. \tag{3.9}$$

If we suppose operator

$$A = 1 - \alpha\Delta t v\Delta,$$

$$\begin{aligned} B_s &= 1 + \alpha\Delta t v\Delta + (\alpha\Delta t v\Delta)^2 + \cdots + (\alpha\Delta t v\Delta)^s \\ &= 1 + \alpha\Delta t v\Delta + (\alpha\Delta t v)^2\Delta^2 + \cdots + (\alpha\Delta t v)^s\Delta^s, \end{aligned}$$

then

$$B_s A = A B_s = 1 - (\alpha\Delta t v)^{s+1}\Delta^{s+1} \tag{3.10}$$

thanks to (3.10), (3.8)

$$\begin{aligned} \Delta\psi^{n+1} &= B_s(f_r^n + \Delta t g_r^n) + (\alpha\Delta t v)^{s+1}\Delta^{s+1}\Delta\psi^{n+1} \\ &= f_r^n + \Delta t[(\alpha v)\Delta f_r^n + g_r^n] + \cdots + (\Delta t)^s[(\alpha v)^s\Delta^s f_r^n + (\alpha v)^{s-1}\Delta^{s-1} g_r^n] \\ &\quad + (\Delta t)^{s+1}[(\alpha v)^s\Delta^s g_r^n + (\alpha v)^{s+1}\Delta^{s+2}\psi^{n+1}] \\ &= f_r^n + \Delta t[(\alpha v)\Delta f_r^n + g_r^n] + \cdots + (\Delta t)^s[(\alpha v)^s\Delta^s f_r^n + (\alpha v)^{s-1}\Delta^{s-1} g_r^n] + \mathcal{O}(\Delta t)^{s+1} \end{aligned} \tag{3.11}$$

Hence, if the stream function ψ^{n+1} is smooth enough, Eq. (3.8) can be reformulated as

$$\begin{aligned} &\Delta\psi^{n+1} = f_r^n + \Delta t[(\alpha v)\Delta f_r^n + g_r^n] + \cdots + (\Delta t)^s[(\alpha v)^s\Delta^s f_r^n + (\alpha v)^{s-1}\Delta^{s-1} g_r^n] + \mathcal{O}(\Delta t)^{s+1}, \\ &\psi^{n+1}|_{\Gamma} = 0, \quad \left.\frac{\partial\psi^{n+1}}{\partial n}\right|_{\Gamma} = 0. \end{aligned} \tag{3.12}$$

Furthermore, the Eqs. (3.12) or (3.8) can be approximated by

$$\begin{aligned} &\Delta\psi^{n+1} = f_r^n + \Delta t[(\alpha v)\Delta f_r^n + g_r^n] + \cdots + (\Delta t)^s[(\alpha v)^s\Delta^s f_r^n + (\alpha v)^{s-1}\Delta^{s-1} g_r^n], \\ &\psi^{n+1}|_{\Gamma} = 0, \quad \left.\frac{\partial\psi^{n+1}}{\partial n}\right|_{\Gamma} = 0, \end{aligned} \tag{3.13}$$

the difference between Eqs. (3.13) and (3.8) is $\mathcal{O}(\Delta t)^{s+1}$.

Thanks to the properties of the Poisson equation, the difference between the solution of (3.13) and the solution of the following equations:

$$\begin{aligned} &\Delta\psi^{n+1} = f_r^n + \Delta t[(\alpha v)\Delta f_r^n + g_r^n] + \cdots + (\Delta t)^s[(\alpha v)^s\Delta^s f_r^n + (\alpha v)^{s-1}\Delta^{s-1} g_r^n], \\ &\left.\frac{\partial\psi^{n+1}}{\partial n}\right|_{\Gamma} = 0 \end{aligned} \tag{3.14}$$

is only a constant. From (3.9) we know that if we have obtained the solution of Eq. (3.14), the velocity $\mathbf{u}^{n+1}$ and the vorticity ω^{n+1} can be obtained approximately. Let $s = r$, Eq. (3.14) reads as

$$\begin{aligned}
&\Delta\psi^{n+1} = f_r^n + \Delta t[(\alpha v)\Delta f_r^n + g_r^n] + \cdots + (\Delta t)^r[(\alpha v)^r \Delta^r f_r^n + (\alpha v)^{r-1}\Delta^{r-1} g_r^n],\\
&\left.\frac{\partial\psi^{n+1}}{\partial n}\right|_\Gamma = 0.
\end{aligned} \tag{3.15}$$

Remark 3.1. The difference between Eqs. (3.6) and (1.3) is $\mathcal{O}(\Delta t)^{r+1}$, the difference between Eqs. (3.6) and (3.13) is $\mathcal{O}(\Delta t)^{s+1}$, so the difference between Eqs. (1.3) and (3.13) is $\mathcal{O}(\Delta t)^{\min\{r+1,s+1\}}$. Thus, for $s = r$, the difference between Eqs. (1.3) and (3.15) is $\mathcal{O}(\Delta t)^{r+1}$.

Particularly, for $r = s = 1$ we obtain the following first-order method:

$$\begin{aligned}
&\Delta\psi^{n+1} = f_1^n + \Delta t[(\alpha v)\Delta f_1^n + g_1^n],\\
&\left.\frac{\partial\psi^{n+1}}{\partial n}\right|_\Gamma = 0
\end{aligned} \tag{3.16}$$

for $r = s = 2$ we obtain the following second-order method:

$$\begin{aligned}
&\Delta\psi^{n+1} = f_2^n + \Delta t[(\alpha v)\Delta f_2^n + g_2^n] + (\Delta t)^2[(\alpha v)^2\Delta^2 f_2^n + (\alpha v)\Delta g_2^n],\\
&\left.\frac{\partial\psi^{n+1}}{\partial n}\right|_\Gamma = 0.
\end{aligned} \tag{3.17}$$

Remark 3.2. If we use the BDF1 method in the time discretization, the first-order method (3.16) can be reformulated as follows:

$$\begin{aligned}
&\frac{\omega^{n+1} - \omega^n}{\Delta t} + \mathbf{u}^n \cdot \nabla\omega^n = v\Delta\omega^n,\\
&\Delta\psi^{n+1} = \omega^{n+1},\\
&\left.\frac{\partial\psi^{n+1}}{\partial n}\right|_\Gamma = 0,
\end{aligned} \tag{3.18}$$

which is the same as the first-order explicit time discretization method of (1.3). At the same time, if we use the BDF2 method in the time discretization, the second-order method (3.17) can be reformulated as

$$\begin{aligned}
&\frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2\Delta t} + (2\mathbf{u}^n \cdot \nabla\omega^n - \mathbf{u}^{n-1} \cdot \nabla\omega^{n-1})\\
&\quad = v(\tfrac{4}{3}\Delta\omega^n - \tfrac{1}{3}\Delta\omega^{n-1}) + \tfrac{2}{3}\Delta t v\Delta(v(\tfrac{4}{3}\Delta\omega^n - \tfrac{1}{3}\Delta\omega^{n-1}) - (2\mathbf{u}^n \cdot \nabla\omega^n - \mathbf{u}^{n-1} \cdot \nabla\omega^{n-1})),\\
&\Delta\psi^{n+1} = \omega^{n+1},\\
&\left.\frac{\partial\psi^{n+1}}{\partial n}\right|_\Gamma = 0,
\end{aligned} \tag{3.19}$$

which is different from the usual explicit time discretization method of (1.3).

Remark 3.3. Both in the first-order method (3.16) and in the second-order method (3.17), we can use any spatial discretization method. The weak formulation of (3.16) and (3.17) can also be used if we use finite element method. Combined with the finite element method, method (3.16) is similar to the simple finite element method which is provided in [20]. On the other hand, if we use the spectral method, radial basis function method or IRBFN method in the spatial discretization, we can directly use the following high-order method:

$$(1 - \alpha\Delta t\, v\Delta)\Delta\psi^{n+1} = f_r^n + \Delta t\, g_r^n,$$
$$\left.\frac{\partial\psi^{n+1}}{\partial n}\right|_{\Gamma} = 0, \tag{3.20}$$

which would be more convenient than (3.15) for $r \geqslant 3$. For $r \leqslant 2$, we suggest that method (3.15) (i.e. (3.16) and (3.17)) should be used.

4. Numerical examples

In this section, we will present some numerical examples to demonstrate the accuracy and efficiency of our meshless method.

We consider a square domain $[0, 1]^2$. We take the exact solution (ω, ψ) of the Navier–Stokes equations

$$\partial_t\omega + (\mathbf{u}\cdot\nabla)\omega = v\Delta\omega + f,$$
$$\Delta\psi = \omega,$$
$$\psi|_{\Gamma} = 0, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\Gamma} = 0$$

to be

$$\psi = -\sin t \sin^2\pi x \sin^2\pi y,$$
$$\omega = -\pi^2 \sin t(\cos 2\pi x + \cos 2\pi y - 2\cos 2\pi x \cos 2\pi y),$$
$$u = \pi \sin t \sin 2\pi y \sin^2\pi x,$$
$$v = -\pi \sin t \sin 2\pi x \sin^2\pi y,$$

and the force

$$\begin{aligned} f = & -\pi^2 \cos t(\cos 2\pi x + \cos 2\pi y - 2\cos 2\pi x \cos 2\pi y) \\ & + \pi^4 \sin^2 t \sin 2\pi x \sin 2\pi y(\cos 2\pi x - \cos 2\pi y) \\ & - 4v\pi^4 \sin t(\cos 2\pi x + \cos 2\pi y - 4\cos 2\pi x \cos 2\pi y). \end{aligned}$$

In the spatial computation we use the following radial basis function:

$$\phi(r) = \left(1 + \left(\frac{r}{\alpha}\right)^2\right)^{\beta},$$

where α, β are parameters. Firstly, we use method (3.16) (note: the force f should be added, and we take the kinetic viscosity to be $v = 1$); the numerical results are (hereafter, the error is computed at the time $t = 1$)

α	β	n	Δt	Error_u	Error_v	Error_ω	Error_ψ
0.5	−1.5	49	0.01	0.0382	0.0381	0.085	0.015
0.5	−1.5	64	0.01	0.0341	0.0344	0.075	0.0132
0.5	−1.5	81	0.01	0.0124	0.0132	0.060	0.0091
0.8	−1.5	81	0.005	0.0086	0.0088	0.042	0.0057
0.8	−1.5	121	0.0025	0.0041	0.0051	0.025	0.0036
0.8	−1.5	121	0.001	0.0036	0.0041	0.014	0.0026

where we take n collocation points in the spatial, Δt is the time step, and error_u, error_v error_ω error_ψ are defined by

$$\text{error}_u = \frac{1}{n}\sum_{i=1}^{n} |(u(1, x_i) - u^{1/\Delta t}(x_i))|, \quad \text{error}_v = \frac{1}{n}\sum_{i=1}^{n} |(v(1, x_i) - v^{1/\Delta t}(x_i))|,$$

$$\text{error}_\omega = \frac{1}{n}\sum_{i=1}^{n} |(\omega(1, x_i) - \omega^{1/\Delta t}(x_i))|, \quad \text{error}_\psi = \frac{1}{n}\sum_{i=1}^{n} |(\psi(1, x_i) - \psi^{1/\Delta t}(x_i))|.$$

From the numerical results we find that method (3.16) is not accurate enough.

Secondly, we carry out the computation using method (3.17) (again the force f should be added, and take the kinetic viscosity to be $v = 1$); the numerical results are

α	β	n	Δt	Error_u	Error_v	Error_ω	Error_ψ
0.8	−1.5	81	0.01	0.0102	0.0111	0.0231	0.0062
0.8	−1.5	81	0.005	0.0089	0.0091	0.0122	0.0051
0.8	−1.5	81	0.0025	0.0047	0.0046	0.0110	0.0038
0.8	−1.5	121	0.001	0.0033	0.0033	0.0105	0.0026
0.8	−1.5	121	0.0005	0.0025	0.0028	0.0089	0.0015

From the numerical results we can see that method (3.17) is better than (3.16).

Remark 4.1. (1) The parameters α, β should be chosen carefully. The collocation matrix might tend to be singular if the parameter α is too large, the accuracy would be limited if the parameter α is too small. At the same time, the parameter β had better not be an integer.

(2) Though we can construct a more accurate numerical method by (3.15), in the practical computation we find that the other more complex methods are not as good as we expected. This is because of the need for high-order partial derivation of the interpolation functions, which would not be accurate enough. In

fact, it is of no use to use the high-order method because the accuracy of the first time step is no more than one order; a rectification of this problem can be found in [41].

5. Conclusions

In this paper, we have provided a new meshless numerical method for the incompressible flows using the radial basis functions. As a meshless method, it can be used in any domain, and can be used in high-dimensional problems. Certainly, the spectral method also can be used in our scheme (3.17) if the domain Ω has good properties. The numerical results indicate our method as being satisfactory.

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